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EE 364 - Fall 2023

Week 1
Supplemental Notes

Logical operations

Compare:
Knowledge: justified true belief.

Q: what is truth?

A: A function, $t: \{\text{statements}\} \longrightarrow \{0, 1\}$.

e.g. P and Q are statements with truth value T / F

$t(P \,\&\, Q) = \min(t(P), t(Q))$

P	Q	P & Q
1	1	1
0	1	0
1	0	0
0	0	0

$t(P \vee Q) = \max(t(P), t(Q))$

P	Q	P ∨ Q
1	1	1
0	1	1
1	0	1
0	0	0

$t(\sim P) = 1 - t(P)$

P	~P
1	0
0	1

"OR" min(1,·)

$t(P \rightarrow Q) = \min(1, \underbrace{1 - t(P)}_{t(\sim P)} + \underbrace{t(Q)}_{t(Q)})$

P	Q	P → Q
1	1	1
0	1	1
1	0	0
0	0	1

$t(P \oplus Q) = |t(P) - t(Q)|$

P	Q	P ⊕ Q
1	1	0
0	1	1
1	0	1
0	0	0

$t(P \leftrightarrow Q) = 1 - |t(P) - t(Q)|$

P	Q	P ↔ Q
1	1	1
0	1	0
1	0	0
0	0	1

Intuition / Implication.

Q: why [false $\rightarrow$ true] is true.

A: by definition. But consider

"trivially true":
(i.e. true for all inputs)

If given number x less than 10 then x less than 100

[i.e. $x < 10 \rightarrow x < 100$]

Case 1: $x = 5$, $5 < 10 \rightarrow 5 < 100$ true $\rightarrow$ true

true

Case 4: $x = 200$, $200 < 10 \rightarrow 200 < 100$ false $\rightarrow$ false

true

Case 2: $x = 50$, $50 < 10 \rightarrow 50 < 100$ false $\rightarrow$ true

true

Case 3: no such x, $x < 10$ and $x \not< 100$.

$\therefore$ true for all inputs.

Theorem (1) $P \,\&\, Q \Longrightarrow P$

(2) $P \Longrightarrow P \vee Q$

Prf: by (binary) truth table.

P	Q	P	$\&$	Q	$\Longrightarrow$	P	P	$\Longrightarrow$	P	$\vee$	Q
1	1	1	1	1	1	1	1	1	1	1	1
0	1	0	0	1	1	0	0	1	0	1	1
1	0	1	0	0	1	1	1	1	1	1	0
0	0	0	0	0	1	0	0	1	0	0	0

QED.

tautology

"equivalent to truth"

$\therefore$ valid since statement is equivalent to "1" for all inputs

$\exists \longleftrightarrow$ "Some"

$\forall \longleftrightarrow$ "All"

$$\bigcap_\alpha A_\alpha \overset{D.M}{=} \Big(\bigcup_\alpha A_\alpha\Big)^c$$

$$x \in \bigcap_\alpha A_\alpha \longrightarrow \forall \alpha,\ x \in A_\alpha$$

Arbitrary A_α, AND $\&, \cap, \forall$.

OR $\vee, \cup, \exists$

$$\bigcup_\alpha A_\alpha \overset{D.M.}{=} \Big(\bigcap_\alpha A_\alpha\Big)^c$$

$$x \in \bigcup_\alpha A_\alpha \longrightarrow \exists \alpha' : x \in A_{\alpha'}$$

Lemma: little theorem used to prove big theorem

Corollary: special case of theorem

Ex: $[(P \vee Q) \to R] \leftrightarrow [P \to (Q \to R)]$.

P	Q	R	$[(P$	$\vee$	$Q)$	$\to$	$R]$	$\leftrightarrow$	$[P$	$\to$	$(Q$	$\to$	$R)]$
1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	1	1	1	1	1	0	1	1	1	1
1	0	1	1	1	0	1	1	1	1	1	0	1	1
0	0	1	0	0	0	1	1	1	0	1	0	1	1
1	1	0	1	1	1	0	0	1	1	0	1	0	0
0	1	0	0	1	1	0	0	$\boxed{0}$	0	1	1	0	0
1	0	0	1	1	0	0	0	$\boxed{0}$	1	1	0	1	0
0	0	0	0	0	0	1	0	1	0	1	0	1	0
			1	2	1	3	1	4	1	3	1	2	1

$\therefore$ False (since the "final" column is not all 1's)

Models

Defn: Models are abstractions of reality with simplified behavior and cause-effect

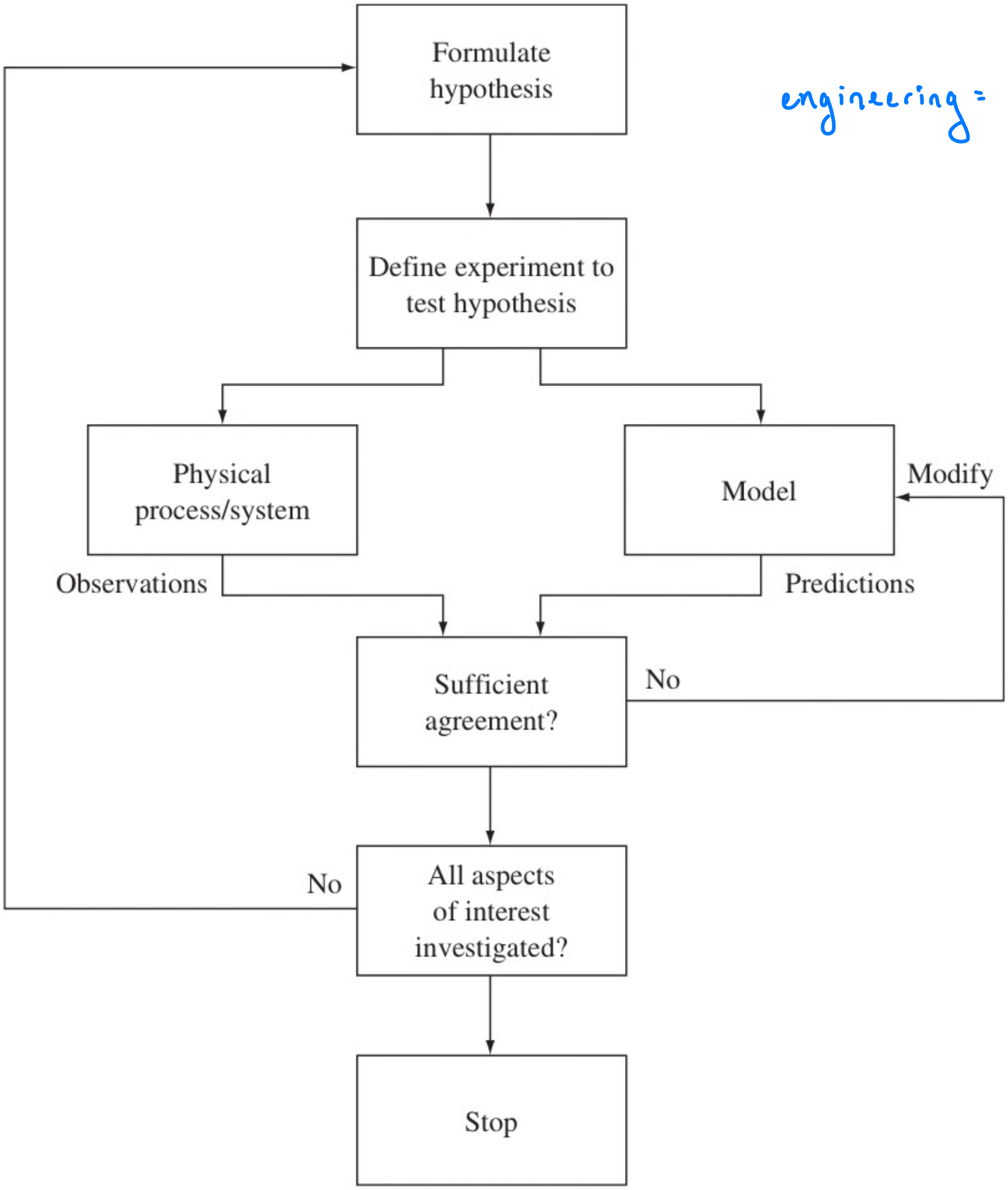
Note: A model is only as good as the modeller and the assumptions that underlie the model.

A useful model explains all relevant aspects ; i.e. what you care about

Ex: Assuming "bell shaped" model for class grades.
(i.e., "the curve")

Goal: explain observed behavior.

Ideal: simple and understandable rules.



engineering = science + money

Each step requires modeller input and expert opinion.

Iterative process continues until model is sufficient
(or run out of time/money) "good enough"

Types of models

- **Deterministic.** Same input produce same output

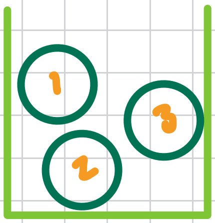
  Ex: $V = iR$.

- **Random** Same input may produce different output.

Defn: A random experiment is an observation that varies in an "unpredictable" way when repeated under the same conditions

"Näive" probability theory.

Ex: 3-balls in urn experiment



1. Shake and randomly select ("random sample")
2. Record #. Return ball ("with replacement")

$\therefore$ Sample space $\Omega = \{1, 2, 3\}$.

Repeat experiment

Outcome varies unpredictably. Cannot infer single outcome given history

Statistical regularity

Regularity is a means to quantify predictability in the long run.

Ex: Weak law of large numbers (WLLN)

each called a "trial"

Average of outcomes from a long sequence of experiments yield approximately the same value

Ex: Central limit theorem (CLT)

Standardized average from a long sequence of experiments follow approximately the same "distribution"

Ex: 3-balls in urn.

Define $N_1(n)$, $N_2(n)$, $N_3(n)$ as number of times you observe 1, 2, 3 after n-flips.

$N_1(n)$, $N_2(n)$, $N_3(n)$ are random.

After you run the experiment and compute $N_1(n)$, $N_2(n)$, $N_3(n)$ they are not random. "Realization" of random variable

Define the relative frequency

$$f_k(n) = \frac{N_k(n)}{n} \quad \text{for } n = 1,2,3$$

Statistical regularity means $f_k(n)$ varies less as n "gets large"

specifically: $\lim_{n \to \infty} f_k(n) = p_k$

constant. called the probability of outcome k.

Problems with this "frequentist" approach:

1. Problems with the limit (in a mathematical sense)

   must $\lim f_x(n)$ always converge? No.

2. Impossible to repeat experiment infinite times

   only finite samples. How to compute P_x?

3. How to operate for experiments you cannot repeat?

   e.g., election outcome

4. How to handle experiments with a continuous sample space. (continuum of outcomes)

   e.g., pick random number in interval $[0,1]$.

$\therefore$ Need a more robust approach but maintains consistency with frequentist approach / intuitive understand of probability.

Axiomatic Probability Theory.

Idea: in random experiments define outcomes and occurance of events as sets

General idea: Probability is about measuring relative size of sets.

Basic structure:

Suppose and experiment yields a random outcome

(1) The set of all possible results Ω is the "sample space"

(2) Elements of the sample space are "outcomes"

(3) Events are a special class of subsets of the sample space called "measurable sets"

(4) The probability of an event is the "size" of the set relative to the whole sample space (Ω)

Axiomatic probability theory uses the language of sets.

Language of sets

$$A \cap B = \{ x \in X : x \in A \ \& \ x \in B \}$$
- set intersection

$$A \cup B = \{ x \in X : x \in A \vee x \in B \}$$
- set union

$$A^c : \{ x \in X : x \notin A \}$$
- set complement

$$A \subset B \longleftrightarrow \forall x : x \in A \longrightarrow x \in B.$$
- set subset

result is a proposition (not set)

$$A - B = A \cap B^c$$
- set difference

$$A = B \longleftrightarrow A \subset B \ \& \ B \subset A.$$
- set equality

$$A \triangle B = (A \cap B^c) \cup (B \cap A^c)$$
- symmetric set difference

De Morgans

$$A \cap B = (A^c \cup B^c)^c \qquad \bigcap_\alpha A_\alpha = \left(\bigcup_\alpha A_\alpha^c \right)^c$$

$$A \cup B = (A^c \cap B^c)^c \qquad \bigcup_\alpha A_\alpha = \left(\bigcap_\alpha A_\alpha^c \right)^c$$

compare: $\min(x,y) = -\max(-x,-y)$
$\max(x,y) = -\min(-x,-y)$

Logical diversion – "how to proof"

* if-then "if p then q", "q if p", "$p \rightarrow q$"

  1. Suppose "if" part is true
  2. Show "then" part is true ("prove") or false (disprove)

* if and only if "p if and only if q", "p iff q", "$p \leftrightarrow q$"

  1. Prove $p \rightarrow q$
  2. Prove $q \rightarrow p$

  must do <u>both</u>

* subset "$X \subset Y$"

  1. Choose arbitrary element $x \in X$.
  2. Show $x \in Y$.

* equality "$X = Y$"

  1. Show $X \subset Y$
  2. Show $Y \subset X$

  must do <u>both</u>

Note: Venn diagrams are <u>NOT</u> proofs. helpful tools use as guidance.

Ex: $A \cap B \subset A \subset A \cup B$.

Pf: Claim 1: $A \cap B \subset A$

$x \in A \cap B$	assump.
$\therefore x \in A$ AND $x \in B$	defn $\cap$
$\therefore x \in A$	Lemma (TT above)
$\therefore A \cap B \subset A$	Defn $\subset$

QED.

Claim 2: $A \subset A \cup B$.

$x \in A$	assump.
$\therefore x \in A$ OR $x \in B$	Lemma (TT above)
$\therefore x \in A \cup B$	Defn $\cup$
$\therefore A \subset A \cup B$	Defn $\subset$

Q: What is probability

A: It is a function. $P: \mathcal{Q} \longrightarrow [0,1]$.

② CAT

- definition of a probability measure

(all possible outcomes)
sample space
all possible events.

Defn: Suppose $(X, \mathcal{Q})$ is a measurable space.
Then $P: \mathcal{Q} \longrightarrow [0,1]$ is a probability measure iff P is CAT.

CA: Countable Additive

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k) \quad \text{if} \quad A_i \cap A_j = \phi \text{ when } i \neq j$$

(A_i and A_j are "mutually exclusive")

T: Total set X gets unit mass:

$$P(X) = 1$$

Defn: $(X, \mathcal{Q}, P)$ is a probability space iff
$(X, \mathcal{Q})$ is a measurable space
and $P: \mathcal{Q} \longrightarrow [0,1]$ is CAT.

Ex: Prove or disprove $[A \cup B = B] \longrightarrow [A \subset B]$.

Prf: method of proof for " $\longrightarrow$ "

1. Assume LHS
2. Determine RHS true / false.

Suppose $A \cup B = B$. [$\therefore$ show $A \subset B$].

P	Q	P $\to$ Q
1	1	1
0	1	1
$\boxed{1}$	$\boxed{0}$	$\boxed{0}$
0	0	1

only row to worry about!

1. pick $x \in A$ — Assump.
2. $\therefore x \in A$ OR $x \in B$ — Defn $\vee$
3. $\therefore x \in A \cup B$ — Defn $\cup$ ← like LHS, that's the key
4. $\therefore x \in B$ — Since $A \cup B = B$ (by assump.)
5. $\therefore A \subset B$ — Defn $\subset$

Ex: $P[\emptyset] = 0$. → CAT, $\therefore$ only know $P[X] = 1$.

Prf: $X = X \cup \emptyset$. X and $\emptyset$ mutually exclusive since $X \cap \emptyset = \emptyset$. and $X \cup \emptyset = X$.

$$\therefore P[X] = P[X \cup \emptyset] \overset{C.A.}{=} P[X] + P[\emptyset]$$

$$\therefore 1 \overset{T}{=} 1 + P[\emptyset]$$

$$\therefore P[\emptyset] = 0.$$

Ex: Monotony $\quad A \subset B \longrightarrow P(A) \leq P(B)$

Prf: Suppose $A \subset B$

$$B = B \cap X = B \cap (A \cap A^c)$$

$$= (B \cap A) \cap (B \cap A^c) \qquad \text{Distributivity (proof req'd)}$$

$$= A \cap (B \cap A^c) \qquad A \subset B \ \text{(by hypo)}$$

$$\therefore P[B] = P[A] + P[B \cap A^c] \qquad (A) \cap (B \cap A^c) = \emptyset.$$

$\therefore$ mut. exclusive + CA

$$\therefore \qquad \geq P[A]$$

Ex: Addition Theorem $\quad P(A) + P(B) = P(A \cup B) - P(A \cap B)$

$$A = A \cap X = A \cap (B \cup B^c)$$

$$B = B \cap X = B \cap (A \cup A^c)$$

$$\therefore P(A) + P(B) = P(\underbrace{A \cap (B \cup B^c)}_{(A \cap B) \cup (A \cap B^c) \ \text{disjoint}}) + P(\underbrace{B \cap (A \cup A^c)}_{(B \cap A) \cup (B \cap A^c) \ \text{disjoint}})$$

$$= P(A \cap B) + \underbrace{P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)}_{= P(A \cup B)} \qquad \text{next page}$$

$$= P(A \cap B) + P(A \cup B)$$

Ex: Addition Theorem (detailed proof)

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

Claim: $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$

$$= \underbrace{(A \cap B^c)}_{\text{prove } P[A] - P[A \cap B]} \cup (A \cap B) \cup \underbrace{(B \cap A^c)}_{\text{prove } P[B] - P[A \cap B]}$$

Lemma: $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$

Claim 1: $A \cup B \subset (A-B) \cup (A \cap B) \cup (B-A)$

1. pick $x \in A \cup B$ — Ass.
2. $\therefore x \in A$ OR $x \in B$ — defn $\cup$.

Case 1: $x \in A$ AND $x \notin B$.

$\therefore x \in A$ AND $x \in B^c$ — defn c

$\therefore x \in A \cap B^c$ — defn $\cap$

$\therefore x \in A - B$ — defn $-$

Case 2: $x \notin A$ AND $x \in B$

$\therefore x \in A^c$ AND $x \in B$. — defn c

$\therefore x \in B \cap A^c$ — defn $\cap$

$\therefore x \in B - A$ — defn $-$

Case 3: $x \in A$ AND $x \in B$.

$\therefore x \in A \cap B$. — defn $\cap$.

But how to justify using cases? propositional logic, next page.

3. $\therefore\ x \in (A-B) \cup (A \cap B) \cup (B-A)$ Lemma

4. $\therefore\ A \cup B \subset (A-B) \cup (A \cap B) \cup (B-A)$ defn $\subset$

QED, claim 1.

Lemma: $P \vee Q \longleftrightarrow (P \,\&\, {\sim}Q) \vee (P \,\&\, Q) \vee ({\sim}P \,\&\, Q)$

P	Q	$P \vee Q \longrightarrow (P \,\&\, {\sim}Q) \vee (P \,\&\, Q) \vee ({\sim}P \,\&\, Q)$
1	1	1 1 1 **1** 1 0 0 1 1 1 **1** 0 0 1
0	1	0 1 1 **1** 0 0 0 0 0 1 **1** 1 1 1
1	0	1 1 0 **1** 1 1 1 1 0 0 **1** 0 0 0
0	0	0 0 0 **1** 0 0 1 0 0 0 **0** 1 0 0

$\therefore$ valid $\therefore$ QED. (Lemma)

Claim 2: $(A-B) \cup (A \cap B) \cup (B-A) \subset A \cup B$

1. pick $x \in (A-B) \cup (A \cap B) \cup (B-A)$. Ass.

2. $\therefore\ x \in (A-B)$ OR $x \in A \cap B$ OR $x \in B-A$ defn $\cup$

Case 1: $x \in A-B$.

1. $\therefore\ x \in A \cap B^c$ defn $-$

2. $\therefore\ x \in A$ AND $x \in B^c$ defn $\cap$

3. $\therefore\ x \in A$ Lemma: $P \,\&\, Q \longrightarrow P$.

4. $\therefore\ x \in A$ OR $x \in B$ Lemma: $P \longrightarrow P \vee Q$.

5. $\therefore\ x \in A \cup B$ defn $\cup$

Case 2: $x \in A \cap B$ ✓

Case 3: $x \in B - A$ ✓

3. $\therefore \; x \in A \cup B.$

4. $\therefore \; (A-B) \cup (A \cap B) \cup (B-A) \subset A \cup B.$ defn $\subset$

QED, Claim 2.

$\therefore \; A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$ defn $=$, Claim 1 & 2.

QED, Lemma.

$$\therefore \; P(A) + P(B) = P(A-B) + P(A \cap B) + P(B-A) + P(A \cap B)$$

$$= P\big((A-B) + P(A \cap B) + P(B-A)\big) + P(A \cap B)$$ disjoint, CA.

$$= P(A \cup B) + P(A \cap B)$$ Lemma.

QED

quod erat demonstrandum
(which was to be demonstrated)